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THE BEHAVIOUR OF A NON-LINEAR HAMILTONIAN SYSTEM WITH ONE DEGREE OF FREEDOM AT THE BOUNDARY OF A PARAMETRIC RESONANCE DOMAIN†

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The non-linear oscillations of a nearly-integrable time-periodic Hamiltonian system with one degree of freedom are studied in the neighbourhood of its equilibrium position. The case when the multipliers of the linearized system are multiple is considered. Using a canonical change of variables the Hamiltonian function is reduced to a simpler form reflecting the resonant nature of the problem under consideration. An approximate result is considered in detail; some of the results are extended to the complete system. A rule is established which enables one to use the nature of the dependence of the non-linear oscillation frequencies on the amplitude in the unperturbed system to distinguish between the boundaries of the parametric resonance domain at which the equilibrium position is stable from those boundaries at which it is unstable. In the unstable case an estimate is given of the size of the equilibrium neighbourhood to which the trajectory of a perturbed system is confined. The existence of stable periodic motions is demonstrated in the neighbourhood of an unstable equilibrium position. The stochastic behaviour of the system is discussed. A number of examples of the application of the general results to specific problems in mechanics are considered.

1. STATEMENT OF THE PROBLEM

Consider a system with one degree of freedom whose motion is described by ordinary Hamiltonian differential equations with a Hamiltonian representable in the form of a series in a small parameter ε $(0 < \varepsilon \ll 1)$

$$H = H^{(0)}(x,y) + \sum_{k=1}^{\infty} \varepsilon^k H^{(k)}(x,y,t) \tag{1.1}$$

Let $x = y = 0$ be the equilibrium position of the system, and suppose that the functions $H^{(k)}$ $(k = 0, 1, 2, \ldots)$ from (1.1) can be represented in the form of series

$$H^{(k)} = H_2^{(k)} + H_3^{(k)} + \ldots + H_m^{(k)} + \ldots$$

where $H_m^{(k)}$ is a form of degree m in x, y. When $k = 0$ the coefficients of these forms are constant, and when $k \geqslant 1$ they are continuous and 2π-periodic in t.

We shall assume that when $\varepsilon = 0$ the equilibrium $x = y = 0$ is stable. We will denote the oscillation frequency of the linearized system by ω $(\omega > 0)$. By an appropriate choice of variables x, y, accomplished, for example, by a Birkhoff transformation [1], the function $H^{(0)}$ can be written in the form

$$H^{(0)} = \tfrac{1}{2}\omega(x^2+y^2) + \tfrac{1}{4}c(x^2+y^2)^2 + O\left((x^2+y^2)^3\right) \tag{1.2}$$

We assume that the quantity c in (1.2) is non-zero.

Supposing that this choice of variables x, y has already been made, we introduce new variables q, p with the help of the canonical transformation $x = \varepsilon^{1/2}q$, $y = \varepsilon^{1/2}p$. We then obtain a new Hamiltonian function of the form

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$$H = \tfrac{1}{2}\omega(q^2+p^2)+\varepsilon\left[\tfrac{1}{4}c(q^2+p^2)^2+H_2^{(1)}(q,p,t)\right]+$$
$$+\varepsilon^{3/2}H_3^{(1)}(q,p,t)+O(\varepsilon^2) \tag{1.3}$$

Without loss of generality we assume that the mean values of the functions $H_2^{(1)}$, $H_3^{(1)}$ with respect to the explicit time variable are zero.

Suppose that the linearized system has a parametric resonance $2\omega \simeq N$, where N is an integer. Using a linear 2π-periodic in t canonical change of variables $q, p \to u, \upsilon$ that is nearly an identity, the Hamiltonian (1.3) can be reduced to the form [2]

$$H = \tfrac{1}{2}\omega(u^2+\upsilon^2)+\varepsilon[\tfrac{1}{4}c(u^2+\upsilon^2)^2+\tfrac{1}{2}(\kappa_1\sin Nt-\kappa_2\cos Nt)(u^2-\upsilon^2)+$$
$$+(\kappa_1\cos Nt+\kappa_2\sin Nt)\,u\upsilon]+\varepsilon^{3/2}H_3^{(1)}(u,\upsilon,t)+O(\varepsilon^2) \tag{1.4}$$

Here

$$\kappa_1 = \frac{1}{2\pi}\int_0^{2\pi}[h_{11}\cos Nt-(h_{02}-h_{20})\sin Nt]dt,\quad \kappa_2 = \frac{1}{2\pi}\int_0^{2\pi}[h_{11}\sin Nt+(h_{02}-h_{20})\cos Nt]dt$$

The quantities $h_{ij} = h_{ij}(t)$ are coefficients of the quadratic form $H_2^{(1)}$ from (1.3): $H_2^{(1)} = h_{20}q^2 + h_{11}qp + h_{02}p^2$.

Let $N - 2\omega = 2\varepsilon\beta$. Changing to the canonically conjugate variables ψ, R by the formulae

$$u = (2R)^{1/2}\sin(\psi+\psi_0+Nt/2),\quad \upsilon = (2R)^{1/2}\cos(\psi+\psi_0+Nt/2)$$
$$\sin 2\psi_0 = \kappa_1\kappa^{-1},\quad \cos 2\psi_0 = \kappa_2\kappa^{-1},\quad \kappa = (\kappa_1^2+\kappa_2^2)^{1/2}$$

we obtain the Hamiltonian

$$H = \varepsilon(-\beta R+\kappa R\cos 2\psi+cR^2)+\varepsilon^{3/2}H_3^{(1)}+O(\varepsilon^2)$$

Here the function $H_3^{(1)}$ is expressed in terms of ψ, R.

We perform yet another canonical transformation $\psi, R \to \theta, \rho$.

$$R = \kappa|c|^{-1}\rho,\quad \psi = \sigma\theta+(1-\sigma)\pi/4\quad (\sigma = \operatorname{sign} c)$$

and change to a new independent variable $\tau = \varepsilon\kappa t$. In the new variables the motion is described by the equations

$$d\theta/d\tau = \partial\gamma/\partial\rho,\quad d\rho/d\tau = -\partial\gamma/\partial\theta \tag{1.5}$$

with a Hamiltonian of the form

$$\gamma = \gamma_0(\theta,\rho)+\varepsilon^{1/2}\gamma_1(\theta,\rho,\tau)+O(\varepsilon) \tag{1.6}$$
$$\gamma_0 = -\mu\rho+\rho\cos 2\theta+\rho^2,\quad \gamma_1 = c\kappa^{-2}H_3^{(1)},\quad \mu = \sigma\beta\kappa^{-1}$$

In the Cartesian canonically conjugate variables $x_1 = (2\rho)^{1/2}\cos\theta, x_2 = (2\rho)^{1/2}\sin\theta$ we have

$$\gamma_0 = \tfrac{1}{2}\left[(\mu-1)x_1^2+(\mu+1)x_2^2\right]-\tfrac{1}{4}(x_1^2+x_2^2)^2 \tag{1.7}$$

The equilibrium position $x = y = 0$ of the original system corresponds to the solution of Eqs (1.5) in which $\rho = 0$. It follows from (1.5)–(1.7) that when the inequality $|\mu| < 1$ is satisfied the equilibrium position is unstable for sufficiently small ε. This inequality specifies the domain of parametric resonance in the first approximation with respect to ε. If, however, $|\mu| > 1$, it follows from known results [3, 4] that for sufficiently small ε the equilibrium position $x = y = 0$ is stable.

This paper investigates the behaviour of the system at the boundary of the domain of parametric resonance, when $\mu = 1$ or -1. Here it is assumed that the mean value of the function γ_1 from (1.6) with respect to the explicit argument τ is zero. This assumption is always satisfied when N is odd. If however N is even this average can be non-zero; and this case requires a special investigation.

2. STABILITY AT THE BOUNDARY OF THE PARAMETRIC RESONANCE DOMAIN

Let $\mu = 1$. Then

$$\gamma_0 = x_2^2 - \tfrac{1}{4}(x_1^2 + x_2^2)^2$$

In y_1, y_2 variables introduced by means of the equalities

$$x_2 = \partial S / \partial x_1, \quad y_1 = \partial S / \partial y_2$$

where

$$S = x_1 y_2 + \tfrac{1}{12} x_1^3 y_2 + \tfrac{1}{8} x_1 y_2^3$$

we have

$$\gamma_0 = y_2^2 - \tfrac{1}{4} y_1^4 + O\left((y_1^2 + y_2^2)^{5/2}\right)$$

When $\varepsilon \neq 0$ we use a real canonical normalizing change of variables $\varepsilon^{1/2}$ that is analytic in $\theta, \rho \to z_1$, z_2 to reduce the Hamiltonian (1.6) the form

$$\gamma_0 = z_2^2 - a_1 / 4 z_1^4 + O\left((z_1^2 + z_2^2)^{5/2}\right) \tag{2.1}$$

The normalizing change of variables is 2π-periodic in t if N is even and 4π-periodic if N is odd. The real coefficient a_1 in (2.1) tends to unity when $\varepsilon \to 0$. The position of equilibrium $x = y = 0$ corresponds to the equilibrium $z_1 = z_2 = 0$ of the transformed system with Hamiltonian (2.1).

We will demonstrate the instability of the equilibrium. To do this we consider the function $V = z_1 z_2$. By virtue of the equations of motion with Hamiltonian (2.1) we obtain the expression

$$dV / d\tau = a_1 z_1^4 + 2 z_2^2 + O\left((z_1^2 + z_2^2)^{5/2}\right)$$

for its derivative.

Because the function $dV/d\tau$ is positive-definite and V is not sign-constant, as opposed to $dV/d\tau$, the first Lyapunov instability theorem [5] implies that the equilibrium $z_1 = z_2 = 0$ is unstable.

This result was obtained previously in [6, 7] by other means.

We will now consider the second boundary of the parametric resonance domain, specified in the first approximation in ε by the equation $\mu = -1$. In this case

$$\gamma_0 = -x_1^2 - \tfrac{1}{4}(x_1^2 + x_2^2)^2$$

and like the case $\mu = 1$ considered previously the Hamiltonian (1.6) can be reduced to the form

$$\gamma = -z_1^2 - \tfrac{1}{4} a_2 z_2^4 + O\left((z_1^2 + z_2^2)^{5/2}\right) \tag{2.2}$$

where, for small ε, the constant coefficient a_2 is nearly equal to unity. According to [7] this implies the stability of the equilibrium $z_1 = z_2 = 0$. This also follows from Section 4 of the present paper (see below).

Thus, as far as stability is concerned, one boundary of the parametric resonance domain is significantly different from the other: the equilibrium position $x = y = 0$ is stable at one and unstable at the other. From this one obtains a simple rule allowing one to determine which of the boundaries of the parametric resonance domain has stability, and which has instability.

The boundaries $\gamma_\pm$ of the domain of parametric resonance in the first approximation in ε are given by the equalities $\omega = N/2 \pm \varepsilon\kappa$. On γ_+ and γ_- we have $\mu = -\sigma$ and $\mu = \sigma$ respectively. If the system is such that when $\varepsilon = 0$ the frequency of small non-linear oscillations decreases as the amplitude increases, i.e. in (1.2) the coefficient $c < 0$ (a system "with a soft restoring force characteristic"), then we have

$\mu = 1$ on γ_+ and $\mu = -1$ on γ_-, and consequently the equilibrium $x = y = 0$ is unstable on γ_+ and stable on γ_-. If however the frequency of small non-linear oscillations increases as the amplitude increases, i.e. the coefficient $c > 0$ in (1.2) (a system with a "hard restoring force characteristic"), then the converse is true: the equilibrium $x = y = 0$ is stable on γ_+ and unstable on γ_-.

3. EXAMPLES

Oscillations of a non-linear conservative system under the action of external periodic forces. Consider the stability of forced periodic oscillations in a system whose motion is described by the equation

$$\ddot{z} + \omega^2 z + \alpha z^2 + \delta z^3 = \varepsilon \sin t \tag{3.1}$$

where ω, α, δ and ε are constants, ω is not an integer, and $0 < \varepsilon \ll 1$.

According to the Poincaré method [8], for sufficiently small ε Eq. (3.1) has a unique solution $z = f(t, \varepsilon)$ that is analytic in ε and 2π-periodic in t, which reduces to the solution $z = 0$ when $\varepsilon = 0$. It can be represented by a series

$$f(t) = \frac{\varepsilon}{\omega^2 - 1} \sin t + \ldots$$

To investigate the stability of this solution we put

$$z = f(t) + \varepsilon^{1/2} \omega^{-1/2} x_*, \quad \dot{x}_* = \omega y_*$$

The equations of the perturbed motion have Hamiltonian form with Hamiltonian

$$H = \tfrac{1}{2}\omega(x_*^2 + y_*^2) + \tfrac{1}{3}\varepsilon^{1/2}\alpha\omega^{-3/2}x_*^3 + \\ + \varepsilon\left[\alpha\omega^{-1}(\omega^2 - 1)^{-1}\sin t\, x_*^2 + \tfrac{1}{4}\delta\omega^{-2}x_*^4\right] + O(\varepsilon^{3/2}) \tag{3.2}$$

Let $2\omega \simeq 1$. The Deprit–Hori method [9] can be used to construct a canonical transformation $x_*, y_* \to u, v$ which reduces the Hamiltonian (3.2) to the form (1.4). Here $N = 1$, and the coefficients c, κ_1 and κ_2 are computed to be

$$c = \frac{9\delta\omega^2 - 10\alpha^2}{24\omega^4}, \quad \kappa_1 = \frac{\alpha}{2\omega(\omega^2 - 1)}, \quad \kappa_2 = 0$$

In the first approximation in ε the boundaries $\gamma_\pm$ of the domain of parametric resonance begin at the point (1/2, 0) in the ω, ε plane and are given by the equations $\omega = 1/2 \pm 4/3\varepsilon\,|\,\alpha\,|$.

From the rule obtained in Section 2 we find that for sufficiently small ε the periodic solution $z = f(t, \varepsilon)$ is stable at the boundary γ_+ if $\delta > 40\alpha^2/9$ and unstable if $\delta < 40\alpha^2/9$. On γ_- we have the opposite situation: we have instability if $\delta > 40\alpha^2/9$ and stability if $\delta < 40\alpha^2/9$.

A pendulum with a vibrating point of suspension. Suppose that the point of suspension of a mathematical pendulum of length l performs harmonic oscillations in the vertical direction with amplitude a and frequency Ω: $z_0(t) = a \cos \Omega t$. The equation of motion of the pendulum has the form

$$d^2q/d\eta^2 + (\omega^2 + \varepsilon\cos\eta)\sin q = 0 \tag{3.3}$$

where q is the angle of inclination of the pendulum to the vertical and $\eta = \Omega t$, $\omega^2 = g/(\Omega^2 l)$, $\varepsilon = a/l$.

The solution $q = 0$ corresponds to the vertical equilibrium position of the pendulum. The linear problem of the stability of this equilibrium reduces to a Mathieu equation. This equation has been thoroughly studied. In particular, the entire ω, ε plane has been subdivided into domains of stability and instability in the linear approximation (the Haynes–Strett diagram) [10]. The domains of instability (domains of parametric resonance) originate at the point $(N/2, 0)$ of the $\varepsilon = 0$ axis. For ω, ε in these domains equilibrium $q = 0$ is also unstable in the strictly non-linear formulation of the problem, which follows from the Lyapunov theorem on stability in the first approximation.

We shall consider the problem of the stability of the equilibrium $q = 0$ for values of ω, and ε that do not lie inside the parametric resonance domains. We shall assume that the quantity ε is small. Equation (3.3) corresponds to the Hamiltonian

$$H = \tfrac{1}{2}p^2 - (\omega^2 + \varepsilon\cos\eta)\cos q \tag{3.4}$$

Note that the series expansion of H has no third-order terms in q and p, and that in the expansion (1.2) for

Hamiltonian (3.4) the coefficient $c = -1/16 \neq 0$ [11]. From this, using stability theory for Hamiltonian systems with one degree of freedom [2] we conclude that inside the linear stability domain the $q = 0$ equilibrium is indeed stable for sufficiently small ε.

It remains to consider the boundaries of the domain of parametric resonance. We take the domain originating at the point $(1/2, 0)$. In the notation of Section 2 its right-hand boundary is the curve γ_+, and the left-hand boundary the curve γ_-. Because $c < 0$, when ε is small the equilibrium $q = 0$ is unstable at γ_+ and stable at γ_-.

For domains of parametric resonance originating at the points $(N/2, 0)$ when $N \geqslant 2$, the result is similar: there is instability at the right-hand boundaries of these domains and stability at the left-hand ones.

4. OSCILLATIONS OF THE UNPERTURBED SYSTEM IN THE CASE $\mu = -1$

If terms of order $\varepsilon^{1/2}$ and above are ignored in the Hamiltonian (1.6), we obtain a system with Hamiltonian γ_0. We shall call this the unperturbed system. We shall investigate non-linear oscillations of the unperturbed system at the boundaries of the parametric resonance domain. We shall first consider the boundary which in the first approximation in ε is specified by $\mu = -1$.

In this case Eqs (1.5) with Hamiltonian $\gamma = \gamma_0$ have the energy integral

$$\rho^2 + 2\rho\cos^2\theta = h \tag{4.1}$$

The phase trajectories in the x_1, x_2 plane are shown on Fig. 1(a). For comparison Fig. 1(b) shows the phase portrait of the linearized unperturbed system.

In the linear system any point of the Ox_2 axis is an equilibrium position; when $x_1 \neq 0$ the trajectories are straight lines parallel to the Ox_2 axis; the origin of coordinates is unstable.

In the non-linear system the trajectories are closed curves along which $\rho = (\cos^4\theta + h)^{1/2} - \cos^2\theta$ $(h > 0)$. Only one equilibrium position exists – the origin of coordinates, corresponding to $h = 0$. When $h > 0$ we have $\rho_2 \leqslant \rho \leqslant \rho_1$, where $\rho_1 = h^{1/2}$, $\rho_2 = (1 + h)^{1/2} - 1$. The differential equations (1.4) are integrated using elliptic functions. Calculations show that

$$\rho = \rho_1 \frac{\rho_2 + \rho_1 \operatorname{sn}^2(u,k) + \rho_2 \operatorname{cn}^2(u,k)}{\rho_1 + \rho_2 \operatorname{sn}^2(u,k) + \rho_1 \operatorname{cn}^2(u,k)} \tag{4.2}$$

$$k^2 = \tfrac{1}{2}(1 - h^{1/2}(1+h)^{-1/2}), \quad u = 2h^{1/4}(1+h)^{1/4}(\tau + \tau_0) \tag{4.3}$$

The quantity τ_0 is an arbitrary constant. For known $\rho(\tau)$ the function $\theta(\tau)$ is found from integral (4.1).

The frequency of non-linear oscillations is given by

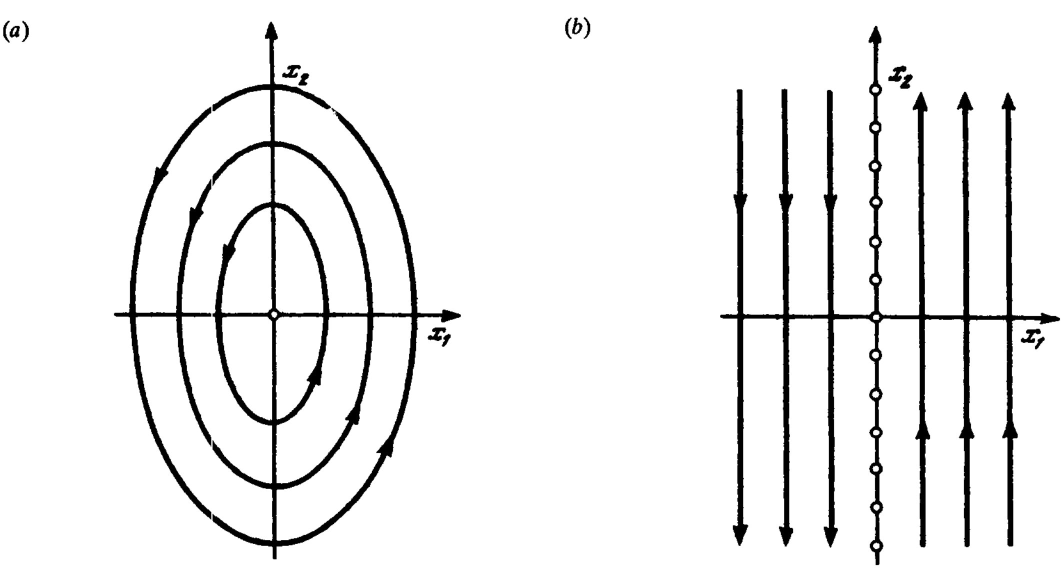

Fig. 1.

$$\omega = \pi h^{1/4}(1+h)^{1/4} K^{-1}(k) \tag{4.4}$$

where $K(k)$ is a complete elliptic integral of the first kind.

The unperturbed Hamiltonian γ_0 can be written in action-angle variables. It is then just a function of the action I: $\gamma_0 = h(I)$. We shall verify the non-degeneracy condition for γ_0. We have

$$\frac{d^2h}{dI^2} = \frac{d\omega}{dI} = \omega\frac{d\omega}{dh} = \frac{\pi^2}{8kh^{1/2}(1+h)K^3}\left[2kK(1+h)^{1/2}(1+2h) + h^{1/2}\frac{dK}{dk}\right] > 0 \tag{4.5}$$

The unperturbed Hamiltonian function is therefore non-degenerate.

When $h \to 0$ the oscillation frequency (4.4) tends to zero

$$\omega \sim bh^{1/4} \quad (b = \pi K^{-1}(\sqrt{2}/2) = 1{,}694) \tag{4.6}$$

Consequently, when $0 < h \ll 1$ (a small neighbourhood of the origin of coordinates $x_1 = x_2 = 0$) we have

$$h \simeq (3bI/4)^{4/3}, \quad d^2h/dI^2 \simeq \tfrac{4}{9}(3b/4)^{4/3} I^{-2/3} \neq 0$$

By virtue of the non-degeneracy of γ_0 and by Moser's theorem on invariant curves {4] we find that the $x = y = 0$ equilibrium position, which is unstable in the linearized problem for the original system, is in fact stable. This result was previously obtained by a somewhat different method in [7].

5. OSCILLATIONS OF THE UNPERTURBED SYSTEM IN THE CASE $\mu = 1$

When $\mu = 1$ the unperturbed system has the integral

$$\rho^2 - 2\rho\sin^2\theta = h \tag{5.1}$$

The phase portrait is shown in Fig. 2(a). The phase portrait of the corresponding linearized system is shown in Fig. 2(b). There, every point on the Ox_1 axis is an equilibrium, and when $x_2 \neq 0$ the phase trajectories are straight lines parallel to the Ox_1 axis; the origin of coordinates is unstable.

As was shown in Section 2, it is also unstable in the perturbed system (1.5).

We consider in detail non-linear oscillations of the unperturbed system. Motion is impossible when

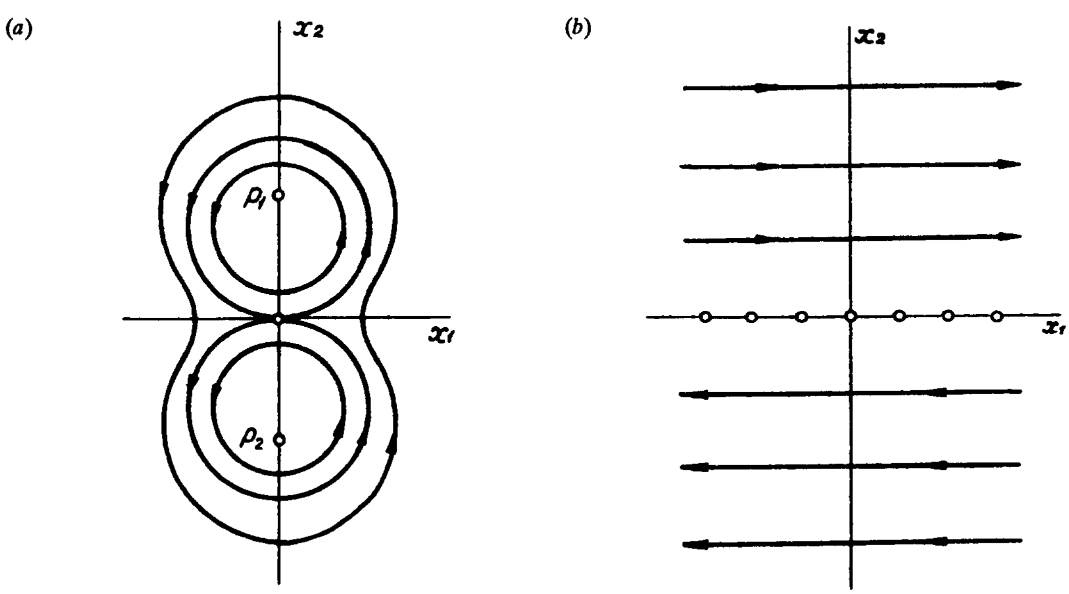

Fig. 2.

$h < -1$. The value $h = -1$ corresponds to equilibrium positions P_1 and P_2 for which $x_1 = 0, x_2 = \pm\sqrt{2}$; the points P_1 and P_2 are centres in the phase plane.

If $-1 < h < 0$ (oscillation domain) the phase trajectories are closed curves encircling the singular points P_1 and P_2. Here $\rho_2 \leqslant \rho \leqslant \rho_1$, where $\rho_1 = 1 + (1+h)^{1/2}$, $\rho_2 = 1 - (1+h)^{1/2}$. The solution $\rho(\tau)$, $\theta(\tau)$ of Eqs (1.5) can be expressed in terms of elliptic functions. We have

$$\rho = \sqrt{-h}\,\frac{1 - d\,\mathrm{cn}(u,k)}{1 + d\,\mathrm{cn}(u,k)} \tag{5.2}$$

where

$$k^2 = \tfrac{1}{2}(1 - \sqrt{-h}),\quad d = \sqrt{1+h}/(1+\sqrt{-h}),\quad u = 2\sqrt{2}(-h)^{1/4}(\tau + \tau_0) \tag{5.3}$$

and the function $\theta(\tau)$ is determined from (5.1) and (5.2). The oscillation frequency is given by

$$\omega = \sqrt{2}\,\pi(-h)^{1/4}K^{-1}(k) \tag{5.4}$$

When $h \to 0$ the frequency tends to zero

$$\omega \simeq \sqrt{2}\,b(-h)^{1/4} \tag{5.5}$$

where b is the quantity in (4.6).

When $h \to -1$ we obtain the frequency of small oscillations in the neighbourhood of P_1 or P_2, equal to $2\sqrt{2}$.

In action-angle variables we have $\gamma_0 = h(I)$. Calculations show that

$$\frac{d^2h}{dI^2} = -\frac{\pi^2}{4k\sqrt{-h}K^3}\left(2kK + \sqrt{-h}\,\frac{dK}{dk}\right) < 0 \tag{5.6}$$

The Hamiltonian of the unperturbed system is therefore non-degenerate in the non-linear oscillation domain in the neighbourhood of the P_1 and P_2 equilibria. In particular, near these equilibria we have

$$h = -1 + 2\sqrt{2}I - \tfrac{5}{4}I^2 + O(I^3)$$

When $h > 0$ (rotation domain) the phase trajectories enclose all three singular points: P_1, P_2 and the origin of coordinates (Fig. 2a). Here $\rho_2 \leqslant \rho \leqslant \rho_1$ where $\rho_1 = 1 + (1+h)^{1/2}$, $\rho_2 = h^{1/2}$, and

$$\begin{gathered}\rho = \rho_2\Lambda_+ / \Lambda_- \\ \Lambda_\pm = (\rho_1 - 1) + (\rho_2 \pm 1)\mathrm{sn}^2(u,k) + (\rho_1 - 1)\mathrm{cn}^2(u,k)\end{gathered} \tag{5.7}$$

The quantities u, k and ω are given by formulae (4.3) and (4.4). By virtue of (4.5) the unperturbed Hamiltonian is non-degenerate. When $h \to 0$ relation (4.6) holds.

The value $h = 0$ corresponds either to the origin of coordinates equilibrium or to the separatrices separating the oscillation and rotation domains on Fig. 2(a). The separatrices are circles $x_1^2 + (x_2 \pm 1)^2 = 1$ from which the point $x_1 = x_2 = 0$ has been removed. On the separatrix lying in the upper half-plane we have (taking $\rho(0) = 2$)

$$\rho = \frac{2}{1+4\tau^2},\quad \sin\theta = \frac{1}{\sqrt{1+4\tau^2}},\quad \cos\theta = -\frac{2\tau}{\sqrt{1+4\tau^2}} \tag{5.8}$$

The corresponding solution for the separatrix lying in the lower half-plane of Fig. 2(a) is obtained from (5.8) by replacing θ by $\pi + \theta$.

6. NON-LINEAR OSCILLATIONS OF THE PERTURBED SYSTEM

We shall explain how the results obtained in Sections 4 and 5 when investigating non-linear oscillations of the unperturbed system with Hamiltonian γ_0 can be extended to the full system with Hamiltonian γ from (1.6).

Outside a sufficiently small neighbourhood of the origin of coordinates and (when $\mu = 1$) the neighbourhoods of the separatrices, the function γ written in action-angle variables is analytic, where, as follows from Sections 4 and 5, its unperturbed part γ_0 is non-degenerate. Hence [12] most closed trajectories in Figs 1(a) and 2(a) generate conditionally-periodic motion when $0 < \varepsilon \ll 1$. The Lebesgue measure of the completion of the set of these generating trajectories is of order [13] $\exp(-c_1\varepsilon^{-1})$ (where $c_1 > 0$ is a constant).

As has been previously remarked, the position of the equilibrium $x = y = 0$ on the $\mu = -1$ boundary of the parametric resonance domain is stable in the full perturbed system, and unstable at the boundary $\mu = 1$. But, as follows from Section 5, in the latter case system trajectories that begin sufficiently close to the origin of coordinates remain in a bounded neighbourhood of the origin (because then for all τ the quantity $\rho(\tau)$ does not exceed a quantity close to 2).

According to Poincaré method for the theory of periodic motion [8], the equilibrium positions P_1 and P_2 for the unperturbed system with Hamiltonian γ_0 when $0 < \varepsilon \ll 1$ become 2π-periodic in t (if N is even) or 4π-periodic in t (if N is odd) motions of the full system, where, in view of the non-degeneracy of γ_0, from Moser's theorem on invariant curves [4] these periodic motions are stable.

When $\mu = 1$ the unperturbed system has four asymptotic trajectories: two of them tend to the origin of coordinates when $\tau \to +\infty$ and two of them do the same when $\tau \to -\infty$ (Fig. 2a). As follows from [6, 14], in a sufficiently small neighbourhood of the origin of coordinates these four asymptotic trajectories also exist in the full system.

If, however, the asymptotic trajectories are pairwise confluent in the unperturbed system, forming separatrices, then in the full system the separatrices in general decouple [12], and there is a stochastic layer in their neighbourhood [15].

Using Chirikov's method [16, 17] we shall obtain an estimate of the width of the stochastic layer. First, following [18], we find the separatrix map for system (1.5) with the full Hamiltonian (1.6). We write the general solution of the unperturbed system in the form

$$\rho = \rho(\tau + \sigma, h), \quad \theta = \theta(\tau + \sigma, h) \tag{6.1}$$

where σ and h are arbitrary constants and h is the constant of the integral (5.1). If we ignore quantities of order ε and above, then for the variables σ and h, which are slowly changing functions of τ in the perturbed problem, we obtain [18] the system of equations

$$dh / d\tau = \varepsilon^{1/2}(\gamma_0, \gamma_1), \quad d\sigma / d\tau = \varepsilon^{1/2}\partial\gamma_1 / \partial h \tag{6.2}$$

where (γ_0, γ_1) is the Poisson bracket. On the right-hand sides of Eqs (6.2) the quantities θ and ρ are expressed in terms of σ, h and τ in accordance with (6.1).

Suppose that h_0 and σ_0 are the values of h and σ at $\tau = 0$. Over a time interval equal to one cycle of motion near the separatrices, the quantities h and σ take the values h_1 and σ_1. The mapping $h_0, \sigma_0 \to h_1, \sigma_1$ is also a separatrix map.

Considering trajectories sufficiently close to a separatrix, one can put $h = 0$ in the right-hand sides of the system of equations (6.2). Here, according to (5.5), the length of a single cycle of the motion near the separatrix is approximately equal to $\sqrt{(2)}\pi b^{-1}|h|^{-1/4}$. As in [18] we find that the separatrix mapping will be approximately given by

$$h_1 = h_0 + \varepsilon^{1/2}G(\sigma_0), \quad \sigma_1 = \sigma_0 + \sqrt{(2)}\pi b^{-1}|h_1|^{-1/4} \tag{6.3}$$

where

$$G(\sigma_0) = \int_{-\infty}^{+\infty}(\gamma_0(\theta, \rho), \gamma_1(\theta, \rho, \tau - \sigma_0))d\tau \tag{6.4}$$

and after calculating the Poisson bracket in the integrand the quantities θ and ρ should be replaced by their expressions from (5.8) (for separatrices lying in the upper half-plane of Fig. 2a).

We represent the function γ_1 from (1.6) in the form

$$\gamma_1 = \rho^{3/2}(f_1 \sin\theta + f_2 \cos\theta + f_3 \sin 3\theta + f_4 \cos 3\theta) \tag{6.5}$$

$$f_i = \sum_{k=1}^{\infty}\left(a_k^{(i)} \sin k\lambda\tau + b_k^{(i)} \cos k\lambda\tau\right)\left(\lambda = (2\varepsilon\kappa)^{-1}\right) \tag{6.6}$$

Calculations then show that

$$G(\sigma_0) = \pi 2^{-5/2}\lambda \sum_{k=1}^{\infty} k\left[4c_k^{(1)} + 2k\lambda(c_k^{(1)} + d_k^{(2)}) - k^2\lambda^2(c_k^{(3)} + d_k^{(4)})\right] e^{-k\lambda/2} \tag{6.7}$$

where

$$c_k^{(i)} = a_k^{(i)} \cos k\lambda\sigma_0 + b_k^{(i)} \sin k\lambda\sigma_0, \quad d_k^{(i)} = -a_k^{(i)} \sin k\lambda\sigma_0 + b_k^{(i)} \cos k\lambda\sigma_0 \tag{6.8}$$

Suppose that the smallest value of k which is actually present in the series (6.7) is q ($q \geqslant 1$). Then for small ε one can write

$$G(\sigma_0) = \chi \sin(q\lambda\sigma_0 + \delta_1)\lambda^p e^{-q\lambda/2}\left(1 + O(\lambda^{-\delta_2})\right) \tag{6.9}$$

where χ, δ_1, δ_2 are some numbers, $\delta_2 \geqslant 1$, and ρ is one of the numbers 1, 2, 3.

If instead of σ we introduce the variable $\alpha = q\lambda\sigma + \delta$, then for small ε the mapping (6.3) can be approximately written in the form

$$h_1 = h_0 + \varepsilon^{1/2}\chi\lambda^p e^{-q\lambda/2} \sin\alpha_0, \quad \alpha_1 = \alpha_0 + \sqrt{2}\pi q b^{-1}\lambda |h_1|^{-1/4} \tag{6.10}$$

At the stationary points of this mapping the value of α is either 0 or π, and

$$h = h_* = \pm \tfrac{1}{4}(q\lambda/(bn))^4 \quad (n = 1, 2, \ldots)$$

Putting

$$h = h_* - 2\sqrt{2}\pi^{-1} b q^{-1}\lambda^{-1} h_* |h_*|^{1/4} P$$

we linearize the mapping (6.10) with respect to P in the neighbourhood of the stationary points. We obtain the standard [17] mapping of the form

$$P_1 = P_0 + K \sin\alpha_0, \quad \alpha_1 = \alpha_0 + P_1$$

where the stochasticity parameter is given by the formula

$$K = -\varepsilon^{1/2} 2^{-3/2}\pi b^{-1}\chi q h_*^{-1} |h_*|^{-1/4} \lambda^{p+1} e^{-q\lambda/2}$$

According to [16, 17] an estimate of the width of the stochastic layer can be obtained from the condition $|K| > 1$. (One must, however, bear in mind that there is no strict justification for this condition, and so the estimate of the width of the stochastic layer here and at the end of the following section does not have a strict mathematical basis.) We have

$$|h_*| < \left(\pi\sqrt{2}|\chi| q/(4b)\right)^{4/5} \varepsilon^{2/5} \lambda^{4(p+1)/5} e^{-2q\lambda/5}$$

From this it follows that the width of the stochastic layer is of the order of

$$\varepsilon^{-c_2} \exp(-c_3\varepsilon^{-1}), \text{ где } c_2 = 2(2p+1)/5, \ c_3 = q/(5\kappa).$$

7. EXAMPLE

On the stochasticity of motions near to eccentric satellite orbits. The plane motion of a solid about its centre of mass in an elliptic orbit is described by the equation [19]

$$(1 + \varepsilon\cos\nu)\frac{d^2\delta}{d\nu^2} - 2\varepsilon\sin\nu\frac{d\delta}{d\nu} + \omega^2 \sin\delta\cos\delta = 2\varepsilon\sin\nu \tag{7.1}$$

where ε is the eccentricity of the orbit of the centre of mass of the body, ν is the true anomaly, $\omega^2 = 3(C - A)/B$, and A, B and C are the principal central moments of inertia of the body (moment of inertia B corresponding to the axis perpendicular to the orbital plane); δ is the angle between the radius-vector of the centre of mass of the body with respect to the gravitationally attractive centre and the axis corresponding to the moment of inertia A.

For a circular orbit ($\varepsilon = 0$) Eq. (7.1) has a solution corresponding to a relative equilibrium of the body. In an elliptical orbit this equilibrium turns into periodic ("eccentric") oscillations. If $\omega \neq 1$ and the orbital eccentricity is sufficiently small, the eccentric oscillations are described [19] by a solution of Eq. (7.1) of the form

$$\delta = \delta_* = \frac{2\varepsilon}{\omega^2 - 1}\sin\nu + \ldots \tag{7.2}$$

which is analytic in ε.

The stability of this solution has been investigated in detail [19, 20]. In particular, it has been shown [19] that if the inequality

$$\tfrac{1}{2} - 3\varepsilon/8 + \ldots < \omega < \tfrac{1}{2} + 3\varepsilon/8 + \ldots \tag{7.3}$$

is satisfied, then for sufficiently small ε the eccentric oscillations are unstable.

We investigate the stability of solution (7.2) and non-linear oscillations in its neighbourhood for values of ω and ε corresponding to the boundaries of the parametric resonance domain (7.3). We put $\omega = \frac{1}{2} + 3\mu/8 + \cdots$. In the notation of Section 2 we have $\mu = 1$ and $\mu = -1$ at the boundaries γ_+ and γ_- respectively.

In a neighbourhood of solution (7.2) we represent Eq. (7.1) in Hamiltonian form. To do this we put

$$\delta = \delta_* + \varepsilon^{1/2}\omega^{-1/2}(1 + \varepsilon\cos\nu)^{-1}\xi, \quad \eta = \omega^{-1}d\xi/d\nu$$

In ξ, η variables the equations of motion have Hamiltonian form. With ξ as the coordinate and η as the momentum the Hamiltonian has the form

$$H = \tfrac{1}{2}\omega(\xi^2 + \eta^2) + \varepsilon(\tfrac{3}{4}\cos\nu\xi^2 - \tfrac{1}{6}\xi^4) + \varepsilon^{3/2}(8\sqrt{2}\sin\nu/9)\xi^3 + O(\varepsilon^2) \tag{7.4}$$

Using the Birkhoff transformation $\xi, \eta \to q, p$ we normalize in (7.4) the terms that do not depend on ν. We then obtain the Hamiltonian (1.3) in which $H_2^{(1)} = \frac{3}{4}\cos\nu q^2$, $H_3^{(1)} = (8\sqrt{(2)}\sin\nu/9)\xi^3$, $c = -\frac{1}{4}$.

We have $c < 0$, and so according to Section 2 the eccentric oscillations (7.2) are unstable at the boundary $\omega = \frac{1}{2} + 3\varepsilon/8 + \cdots$ for sufficiently small ε, while at the boundary $\omega = \frac{1}{2} + 3\varepsilon/8 + \cdots$ they are stable.

We then transform the Hamiltonian (1.3) to the form (1.4) and obtain $\kappa_1 = 0$, $\kappa_2 = -\kappa = -3/8$. We then make the canonical change of variables $u, \upsilon \to \theta, \rho$ using the formulae

$$u = (3\rho)^{1/2}\sin(\theta - \nu/2), \quad \upsilon = -(3\rho)^{1/2}\cos(\theta - \nu/2)$$

and changing to the new independent variable $\tau = 3\varepsilon\nu/8$ we arrive at the Hamiltonian (1.6) in which γ_1 has the form (6.5), (6.6) where $\lambda = 4/(3\varepsilon)$, and

$$f_1 = -16\sqrt{6}(\sin\lambda\tau + \sin 3\lambda\tau)/9, \quad f_2 = 16\sqrt{6}(\cos\lambda\tau - \cos 3\lambda\tau)/9$$

$$f_3 = -16\sqrt{6}(\sin\lambda\tau - \sin 5\lambda\tau)/27, \quad f_4 = -16\sqrt{6}(\cos\lambda\tau - \cos 5\lambda\tau)/27$$

The nature of the non-linear satellite oscillations near to its motion (7.2) can then be described as in Sections 4–6. In particular, we find that when estimating the width of the stochastic layer for the boundary $\omega = 1/2 + 3\varepsilon/8 + \cdots$, the constants χ, q, p, δ_1 and δ_2 on the right-hand side of expression (6.9) take the numerical values $8\pi\sqrt{(3)}/27$, 1, 3, $\pi/2$ and 1, respectively. For small ε the width of the stochastic layer is of the order of $\varepsilon^{-14/5}\exp(-8/(15\varepsilon))$.

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